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RANK-ONE WEAK MIXING FOR NONSINGULAR TRANSFORMATIONS

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ABSTRACT

We construct rank-one infinite measure preserving transformations satisfying each of the following dynamical properties: (1) Continuous L^∞ spectrum, conservative k-fold cartesian products but nonergodic cartesian square; (2) ergodic k-fold cartesian products; (3) nonconservative cartesian square. We show how to modify the construction of (1) to obtain type III_λ transformations with similar properties.

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Introduction

We construct rank-one infinite measure preserving and nonsingular transformations with various dynamical properties. The constructions are by cutting and stacking and of a different nature than infinite measure preserving random walks or Markov shifts that have been used to examine similar properties.

Let (X, μ) be a measure space isomorphic to the unit interval or the real line with Lebesgue measure. A **transformation** is a bimeasurable bijection of X. A transformation T is **measure preserving** if for all measurable sets A, $\mu(A) = \mu(T^{-1}A)$; it is **nonsingular** if for all measurable sets A, $\mu(A) = 0$ if and only if $\mu(T^{-1}A) = 0$. A nonsingular transformation is **conservative** if for all sets A with $\mu(A) > 0$ there exists $n > 0$ such that $\mu(A \cap T^{-n}A) > 0$. T is **ergodic** if for all measurable A, $T^{-1}A = A$ implies $\mu(A) = 0$ or $\mu(A^c) = 0$. One can show that, in a nonatomic space, an (invertible) ergodic transformation is conservative. The property that for all sets of positive measure A and B there is an integer $n > 0$ such that $\mu(A \cap T^n B) > 0$ is equivalent to T being conservative ergodic. Given a transformation T, an L^∞ **eigenvalue** is a complex number λ such that for some non-null function f in L^∞, $f(Tx) = \lambda f(x)$ a.e. Since the L^∞ norms of f and $f \circ T$ are equal, eigenvalues must have modulus 1. If T is ergodic then $|f|$ must be constant a.e. T has **continuous L^∞ spectrum** if it is ergodic and its only L^∞ eigenvalue is 1. The notion of weak mixing for nonsingular transformations is studied in [ALW] and is shown to be equivalent to continuous L^∞ spectrum.

We describe briefly our "cutting and stacking" constructions [F1]. A **level** is a finite left-closed right-open interval. A **column** C of **height** h consists of h disjoint levels $C = \{B_0, B_1, \ldots, B_{h-1}\}$. We think of a column C as partially defining a transformation on each level except the top. The transformation is defined on level B_i by the unique affine map that sends the interval B_i onto the interval B_{i+1}. (The intervals in a column need not be of the same length in the nonsingular case, and when they all have the same length the transformation is measure preserving.) A cutting and stacking rank-one construction consists of a sequence of columns C_n such that C_{n+1} has been obtained from C_n by cutting each level of C_n into r_n subintervals where the ratios of the lengths of the subintervals to the level is the same for all levels in a given column, and by possibly placing additional sublevels, called **spacers** above the top level of C_n. Furthermore, the union of the levels in all the columns generates the Borel sets.

(In the measure preserving case the ratios are determined by the number of cuts r_n; in the nonsingular case the ratios must be specified.) It follows by similar arguments to the finite measure preserving case that these transformations are conservative ergodic [F1].

Our first example is an infinite measure preserving rank-one tranformation with continuous L^∞ spectrum but nonergodic cartesian square. As is well known, for finite measure preserving transformations, continuous spectrum is equivalent to ergodic cartesian square. In general, ergodic cartesian square implies continuous spectrum. The first example showing the converse of this statement is not true for infinite measure was given by Aaronson, Lin and Weiss in [ALW]; however our example is of a different nature, and it generalizes to give examples with the same phenomena of type III_λ nonsingular transformations. The example in [ALW] is an infinite measure preserving Markov shift (the square of a random walk on the integers) that fails to be ergodic when it fails to be conservative. (It follows from [P] that conservative infinite measure preserving Markov shifts whose underlying stochastic matrices are recurrent and aperiodic (hence irreducible) are ergodic.) In our case we obtain nonergodic cartesian square but with all k-fold cartesian products conservative. We then construct infinite measure preserving rank-one examples where all k-fold products are ergodic and where the 2-fold product is not conservative. We also note that infinite measure preserving cutting and stacking rank-one constructions must have zero Krengel entropy (this is the case since the induced transformation on any level must be rank-one and of course finite measure preserving), while random walks on the integers have infinite entropy [F3].

By Proposion 4.3 of [ALW], the examples of Kakutani and Parry [KP] of ergodic index 3 also yield infinite measure preserving transformations with continuous L^∞ spectrum but nonergodic cartesian square. However, these examples also give Markov shifts with nonconservative cartesian square.

The Radon–Nikodym derivative of a nonsingular transformation T is defined to be $d\mu \circ T/d\mu$ and is denoted by ω_T. Given a nonsingular transformation T on (X,μ), with μ σ-finite, the **ratio set** of T, $r(T)$, is defined to be the set of nonnegavite real numbers t such that for all $\varepsilon > 0$ and all measurable sets A there exists $n > 0$ such that

$$\mu(A \cap T^{-n}A \cap \{x: \omega_{T^n}(x) \in N_\varepsilon(t)\}) > 0,$$

where $N_\varepsilon(t)$ is an ε neighborhood of t.

The set $r(T) \setminus \{0\}$ is a closed multiplicative subgroup of the reals and if $0 \in r(T)$ then T admits no equivalent σ-finite invariant measure. We say T is **type** $\mathbf{III}_\lambda$, $0 < \lambda < 1$, if its ratio set is $\{\lambda^n : n \in Z\} \cup \{0\}$. For background material we refer to [F1], [HO] and [KW].

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1. Continuous spectrum with conservative nonergodic cartesian square

We define our transformation S by giving the sequence of columns C_n. Let $C_0 = [0,1)$ and $h_0 = 1$. Given the column C_n of height h_n, we obtain C_{n+1} by first cutting C_n into two subcolumns of equal width and placing $2h_n + 1$ spacers on top of the right-hand subcolumn, to obtain a new right subcolumn. Then stack the (new) right subcolumn on top of the left subcolumn, (i.e., S takes the top level of the left subcolumn to the bottom level of the right subcolumn). Thus $h_{n+1} = 4h_n + 1$. It is clear that S is infinite measure preserving.

PROPOSITION 1.1: *S has continuous L^∞ spectrum.*

Proof: Let $f \in L^\infty$ be such that $f(Sx) = \lambda f(x)$ a.e. We may assume that $\|f\| = 1$. Given $\epsilon > 0$ there exists a set A of positive measure such that for all $x, y \in A$,

$$\left| \frac{f(y)}{f(x)} - 1 \right| < \epsilon.$$

Since levels generate the measurable sets, there exists a level I of C_n, for some n, such that

$$\mu(A \cap I) > \tfrac{11}{12}\mu(I).$$

Partition I into four subintervals of equal length; since A covers more than 2/3 of each subinterval of I, we can choose $x \in A \cap I$ such that $S^{h_n}x$ and $S^{4h_n+1}x$ are in $A \cap I$. By considering $f(S^{h_n}x)/f(x)$ and $f(S^{4h_n+1}x)/f(x)$ we get:

$$|\lambda^{h_n} - 1| < \epsilon \quad \text{and} \quad |\lambda^{4h_n+1} - 1| < \epsilon.$$

The first inequality gives that $|\lambda^{4h_n} - 1| < 4\epsilon$. Therefore

$$|\lambda - 1| = |\lambda^{4h_n+1} - \lambda^{4h_n}| < 5\epsilon.$$

(The argument only uses that the number of spacers is $kh_n + 1$, for some $k > 0$.) ∎

We will use the notion of partial rigidity and bounded Radon–Nikodym derivative to show our examples have conservative k-fold products for all positive integers k. Partial rigidity was introduced in [F2] for finite measure preserving transformations. We say a nonsingular transformation S is **partially rigid** if there exists $\eta > 0$, an increasing sequence r_n, and a constant $R > 0$ such that for all sets A of finite measure

$$\liminf_{n\to\infty} \mu(S^{r_n} A \cap A) \geq \eta\mu(A) \quad \text{and} \tag{1}$$

$$\omega_{S^{r_n}}(x) < R \quad \text{a.e.} \tag{2}$$

A. del Junco and the third-named author have defined rigidity for nonsingular transformations by requiring in addition that the Radon–Nikodym derivatives converge to 1 in L^1. This notion will appear elsewhere and is not necessary in this paper.

LEMMA 1.2: *Let S be a nonsingular transformation satisfying condition* (1) *for all sets A in a dense algebra of sets of finite measure, and condition* (2) *above. Then S is partially rigid along the same sequence r_n.*

Proof: Let E be a set of finite measure. Choose a set G of finite measure which is in the dense algebra and such that $\mu(E\Delta G) < \eta/3R$. After applying the triangle inequality we get

$$\begin{aligned}\mu([S^{r_n}E \cap E]\Delta[S^{r_n}G \cap G]) &\leq \mu(S^{r_n}E \cap (E\Delta G)) + \mu(S^{r_n}(E\Delta G) \cap G)\\ &\leq \mu(E\Delta G) + \mu(S^{r_n}(E\Delta G)) = \mu(E\Delta G) + \int_{E\Delta G} \omega_{S^{r_n}}(x)\\ &\leq (1+R)\mu(E\Delta G).\end{aligned}$$

Finally, since $S^{r_n}E \cap E \supset (S^{r_n}G \cap G) \setminus ([S^{r_n}E \cap E]\Delta[S^{r_n}G \cap G])$, we have

$$\mu(S^{r_n}E \cap E) \geq \eta\mu(G) - (1+R)\mu(E\Delta G).$$

By taking the limit infimum on the left hand side and choosing a sequence of sets G arbitrarily close to E we obtain the desired result. ∎

PROPOSITION 1.3: *If nonsingular transformations S and T are partially rigid along the same sequence r_n, then $S \times T$ is partially rigid along r_n.*

Proof: Suppose S and T are partially rigid on the sequence r_n; we can assume the constants η and R are the same for S and T. It is not difficult to check that $S \times T$ satisfies the condition of partial rigidity for finite unions of disjoint rectangles along the sequence r_n and with constants η^2 and R^2. Now apply Lemma 1.2. ∎

COROLLARY 1.4: *If a nonsingular transformation S is partially rigid then, for all $k > 1$, its k-fold cartesian product is conservative.*

Proof: We first show that if S is partially rigid then it is conservative. If A has positive measure choose $A' \subset A$ of finite positive measure, then by partial rigidity there will exist some n such that $\mu(S^{r_n}(A') \cap A') > \eta\mu(A')/2 > 0$.

Now the corollary follows by repeated applications of Proposition 1.3. ∎

THEOREM 1.5: *The transformation S has continuous L^∞ spectrum, nonergodic cartesian square and all k-fold cartesian products conservative.*

Proof: First we show that S is partially rigid. Let I be a level in column C_n. From the construction of S, $\mu(S^{h_n} I \cap I) = \frac{1}{2}\mu(I)$. By approximating sets by disjoint unions of levels we get that S is partially rigid.

We now show that $S \times S$ is not ergodic. We find A and B of positive measure such that for all $n > 0$,

$$\mu \times \mu((S \times S)^n (A \times A) \cap (A \times B)) = 0.$$

Let B be the top level of C_1 and $A = S^{-1}B$. It is convenient to view the levels of C_n cycling around a circle. Given n let R_n be the cycle on C_n, i.e. R_n is the transformation on C_n that maps each level to the one above it, and the top to the bottom. (R_n agrees with S on all levels of C_n except the top level.) For $L = A$ or $L = B$ define

$$I_n(A, L) = \{i \colon 0 \leq i < h_n, \mu(R_n^i A \cap L) > 0\},$$

and

$$I_n = I_n(A, A) \cap I_n(A, B).$$

We show by induction that $I_n = \emptyset$. It is clear that $I_1 = \emptyset$. Assume that $I_n = \emptyset$. Each of A and B appears as a union of levels in both C_n and C_{n+1}. From the construction of S we obtain the following inclusion for $L = A$ or $L = B$:

$$I_{n+1}(A, L) \subset I_n(A, L) \cup (I_n(A, L) + h_n) \cup (I_n(A, L) + 2h_n + 1) \cup (I_n(A, L) + 3h_n + 1).$$

Note that $R_{n+1}^{2h_n}A$ and $R_{n+1}^{2h_n+1}A$ are contained in the spacers placed on the right subcolumn of C_n. Thus

$$\begin{aligned} I_{n+1} &= I_{n+1}(A,A) \cap I_{n+1}(A,B) \\ &\subset I_n(A,A) \cap I_n(A,B) \\ &\cup (I_n(A,A)+h_n) \cap (I_n(A,B)+h_n) \\ &\cup (I_n(A,A)+2h_n+1) \cap (I_n(A,B)+2h_n+1) \\ &\cup (I_n(A,A)+3h_n+1) \cap (I_n(A,B)+3h_n+1). \end{aligned}$$

By induction each row in the above expression is the empty set; therefore $I_{n+1} = \emptyset$. For all i there exists $n > 0$ such that $S^iA = R_n^iA$. Therefore for all i, $\mu \times \mu((S \times S)^i(A \times A) \cap (A \times B)) = 0$. ∎

2. Rank-one infinite measure with ergodic k-fold cartesian products

We define our transformation T. Let $C_0 = [0,1)$. Given column C_n with height h_n, we obtain C_{n+1} by cutting C_n into three subcolumns of equal width, placing one spacer above the middle subcolumn, placing $3h_n + 1$ spacers above the rightmost subcolumn, and then stacking the (new) middle subcolumn on the left subcolumn to make an intermediate subcolumn, and then the (new) right subcolumn on top of the intermediate subcolumn. This can be thought of as an infinite measure preserving Chacon transformation.

Define $H_{n,0} = 0$, for $n \in Z^+$ and $H_{n,\ell} = \sum_{i=n}^{n+\ell-1} h_i$ for $n \in Z^+$ and $\ell \geq 2$.

Definition: We say levels A and B in C_n are $|\ell|$ *apart* if B is the $i+\ell$ level in C_n when A is the i level.

The following lemma utilizes the role of the single spacer placed on the middle subcolumn.

LEMMA 2.1: *Let ℓ and n be positive integers. If I and J are levels in C_n which are less than ℓ levels apart, and where I is above J, then*

$$\mu(T^{H_{n,\ell}}I \cap J) \geq \frac{1}{3^\ell}\mu(J).$$

Proof: Fix a positive integer n. Let I be a level in C_n which is contained in the top $(h_n - \ell)$ levels of C_n. We will prove inductively on ℓ that there exist levels $I_j = I_j(\ell)$ for $0 \leq j \leq \ell - 1$ in $C_{n+\ell}$ such that $I_j \subset T^{H_{n,\ell}}I \cap T^{-j}I$. This is sufficient to prove the lemma since

$$\mu(I_j) = \frac{1}{3^\ell}\mu(I).$$

For the primary case we have $H_{n,1} = h_n$. The absence of a spacer on the first subcolumn of C_n and the placement of a single spacer on the middle subcolumn ensure that $T^{h_n} I$ contains two levels in C_{n+1} with one contained in I and the other contained in $T^{-1} I$.

Now suppose there exist levels $I_j(r-1)$ in C_{n+r-1} such that $I_j(r-1) \subset T^{H_{n,r-1}} I \cap T^{-j} I$ for $0 \le j \le r-2$. Since $I_j(r-1)$ is a level in C_{n+r-1} for $0 \le j \le r-2$, then there exist levels $I_j(r)$ in C_{n+r} for $0 \le j \le r-2$ such that

$$I_j(r) \subset I_j(r-1) \cap T^{h_{n+r}} I_j(r-1) \subset T^{H_{n,r}} I \cap T^{-j} I.$$

Finally there exists a level $I_{r-1}(r)$ in C_n such that

$$I_{r-1}(r) \subset T^{-1} I_{r-2}(r-1) \cap T^{h_{n+r}} I_{r-2}(r-1) \subset T^{H_{n,r}} I \cap T^{-(r-1)} I. \qquad \blacksquare$$

THEOREM 2.2: *The k-fold cartesian product of T is conservative ergodic.*

Proof: Let ν denote the k-fold product of Lebesgue measure and let S denote the k-fold product of T. Also let E and F be subsets of the product space satisfying $\nu(E) > 0$ and $\nu(F) > 0$. We will exhibit positive integers n and ℓ such that $\nu(S^{H_{n,\ell}} E \cap F) > 0$. Choose levels A_i, B_i for $1 \le i \le k$ in some column C_{m-1} so that

$$\nu(E \cap A) > (\tfrac{17}{18})\nu(A) \quad \text{and} \quad \nu(F \cap B) > (\tfrac{17}{18})\nu(B)$$

where $A = A_1 \times A_2 \times \cdots \times A_k$ and $B = B_1 \times B_2 \times \cdots \times B_k$.

For each i choose the top copy of A_i in C_m and the bottom copy of B_i in C_m and rename them A_i and B_i, respectively. Then A_i will be above B_i in C_m, for $1 \le i \le k$, and $A = A_1 \times A_2 \times \cdots \times A_k$ and $B = B_1 \times B_2 \times \cdots \times B_k$ will satisfy

$$\nu(E \cap A) > (\tfrac{5}{6})\nu(A) \quad \text{and} \quad \nu(F \cap B) > (\tfrac{5}{6})\nu(B). \tag{1}$$

For convenience let E and F denote $E \cap A$ and $F \cap B$, respectively. Let $\ell = h_m$. Thus A_i and B_i are less than ℓ apart for each i. Denote

$$\delta = \frac{1}{3^\ell}.$$

Let $n \ge m$ and label the copies of C_m in C_n from 1 to 3^{n-m}.

Define $V = \{(v_1, \ldots, v_k)\colon v_i = 1, 2, \ldots, 3^{n-m}\}$. Given $u = (u_1, \ldots, u_k) \in V$ let $E_u = I_1 \times \cdots \times I_k$ where $I_j \subset A_j$ is also contained in the u_j copy of C_m in

C_n. Thus $A = \bigcup_{v\in V} E_v$ and $B = \bigcup_{v\in V} F_v$. Pick n large enough so there exist subsets U' and V' of V such that

$$E' = \bigcup_{v\in U'} E_v$$

and

$$F' = \bigcup_{v\in V'} F_v$$

satisfy

$$\nu(E'\triangle E) < \tfrac{1}{18}\delta^k\nu(A) \quad \text{and} \quad \nu(F'\triangle F) < \tfrac{1}{18}\delta^k\nu(B). \tag{2}$$

Thus (1) and (2) give

$$\nu(E'\triangle A) < \tfrac{1}{3}\nu(A) \quad \text{and} \quad \nu(F'\triangle B) < \tfrac{1}{3}\nu(B). \tag{3}$$

Let $U'' = \{v \in U'\colon \nu(E_v \setminus E) < (\frac{1}{3}\delta^k)\nu(E_v)\}$ and define $E'' = \bigcup_{v\in U''} E_v$. Choose V'' and F'' similarly.

Then

$$\begin{aligned}\tfrac{1}{3}\delta^k\nu(E'\setminus E'') &= \sum_{v\subset U'\setminus U''} \tfrac{1}{3}\delta^k\nu(E_v)\\ &\le \sum_{v\subset U'\setminus U''} \nu(E_v\setminus E)\\ &\le \nu(E'\setminus E).\end{aligned}$$

By the triangle inequality,

$$\nu(E''\triangle A) < \tfrac{1}{2}\nu(A).$$

Similarly

$$\nu(F''\triangle B) < \tfrac{1}{2}\nu(B).$$

Therefore $U'' \cap V'' \neq \emptyset$.

Let $u \in U''\cap V''$. Thus by Lemma 2.1

$$\nu(S^{H_{n,\ell}}E_u \cap F_u) \ge \delta^k\nu(F_u).$$

If we let $W = (S^{H_{n,\ell}}E\cap F)$ and $Z = (S^{H_{n,\ell}}E_u\cap F_u)$, we get

$$\begin{aligned}\nu(W) &\ge \nu(Z) - \nu(Z\setminus W)\\ &\ge \nu(S^{H_{n,\ell}}E_u\cap F_u) - \nu(E_u\setminus E) - \nu(F_u\setminus F).\end{aligned}$$

Therefore

$$\begin{aligned}\nu(W) &\ge \delta^k\nu(F_u) - \tfrac{1}{3}\delta^k\nu(E_u) - \tfrac{1}{3}\delta^k\nu(F_u)\\ &= \tfrac{1}{3}\delta^k\nu(F_u). \qquad \blacksquare\end{aligned}$$

3. Rank-one with nonconservative cartesian square

We construct infinite measure preserving rank-one transformations T with $T\times T$ not conservative. Let r_n be a sequence of integers with $r_n \geq 2$. Let $C_1 = [0,1)$ and obtain C_{n+1} from C_n in the following way. Cut C_n into r_n subcolumns of equal width, place $(2^{r_n-i}-1)h_n$ spacers on the ith subcolumn for $1 \leq i \leq r_n - 1$, and place $[r_n h_n + \sum_{i=1}^{r_n-1}(2^{r_n-i}-1)h_n]$ spacers on the last subcolumn. (The number of spacers on the last subcolumn is chosen so that they form half of the levels in C_{n+1}.) Then stack from left to right to form C_{n+1} of height h_{n+1}.

THEOREM 3.1: *If $\sum_{n=1}^{\infty}(1/r_n) < \infty$ then $T\times T$ is not conservative.*

Proof: Choose $n \in \mathbf{Z}^+$ such that $\sum_{k=n}^{\infty}(1/r_k) < 1$. Let A be the top level of C_n. Let $C_{k,\ell}$ denote the ℓth subcolumn of C_k and denote $A_{k,\ell} = A \cap C_{k,\ell}$. Note that for $k \geq n$,

$$\mu\times\mu(A_{k,\ell}\times A_{k,\ell}) = \frac{1}{r_k^2}\mu(A)^2.$$

Hence

$$\mu\times\mu\left(\bigcup_{\ell=1}^{r_k}(A_{k,\ell}\times A_{k,\ell})\right) = \frac{1}{r_k}\mu(A)^2.$$

Define our exceptional set by

$$E = A\times A\setminus\Big[\bigcup_{k=n}^{\infty}\bigcup_{\ell=1}^{r_k}(A_{k,\ell}\times A_{k,\ell})\Big].$$

Thus $\mu\times\mu(E) \geq \mu(A)^2(1-\sum_{k=n}^{\infty}(1/r_k)) > 0$. We will prove inductively on k that

$$\mu\times\mu((T\times T)^i E\cap E) = 0$$

for $1 \leq i \leq h_k$.

Since A is a single level in C_n and we place spacers on each subcolumn $C_{n,\ell}$ then $\mu\times\mu((T\times T)^i E\cap E) = 0$ for $1 \leq i \leq h_n$.

Now suppose $\mu\times\mu((T\times T)^i E\cap E) = 0$ for $1\leq i\leq h_k$, and let $h_k \leq i \leq h_{k+1}$.

We first claim that for $\ell \neq m$, $\mu\times\mu[(T\times T)^i(C_{k,\ell}\times C_{k,m})\cap(C_k\times C_k)] = 0$. Since we place $(2^{r_k-\ell}-1)h_k$ spacers on $C_{k,\ell}$ we have that $\mu(T^i C_{k,\ell}\cap C_k) = 0$ for $h_k \leq i \leq (2^{r_k-\ell}-1)h_k$. The number of iterations for the bottom level of subcolumn $C_{k,\ell}$ to reach the first spacer above C_{k,r_k} equals

$$(r_k-(\ell-1))h_k + \sum_{j=\ell}^{r_k-1}(2^{r_k-j}-1)h_k = \sum_{j=0}^{r_k-\ell}2^j h_k = (2^{r_k-\ell+1}-1)h_k.$$

Hence $\mu(T^i C_{k,\ell} \cap C_k) = 0$ for $(2^{r_k-\ell+1} - 1)h_k \leq i \leq h_{k+1}$. Therefore the sets $I_\ell = \{h_k < i \leq h_{k+1}\colon \mu[T^i C_{k,\ell} \cap C_k] > 0\}$ are disjoint for different ℓ and hence

$$\mu \times \mu[(T \times T)^i (C_{k,\ell} \times C_{k,m}) \cap (C_k \times C_k)] = 0$$

for $\ell \neq m$. (Note that for $i > h_k$ the last subcolumn is moved into the spacers.) Finally, since $\mu \times \mu[E \cap (C_{k,\ell} \times C_{k,\ell})] = 0$ then

$$\mu \times \mu[(T \times T)^i E \cap E] = 0$$

for $1 \leq i \leq h_{k+1}$. ∎

4. Rank-one type III examples

We construct type III$_\lambda$ examples, $0 < \lambda < 1$, with the properties of the example of section 1. Let $C_0 = [0, 1)$ and $h_0 = 1$. To obtain column C_1 first cut C_0 into two subintervals of lengths $\lambda/(1 + \lambda)$ and $1/(1 + \lambda)$ and place the right subinterval on top of the left to obtain an intermediate column C_1'. Then place a copy of C_1' on top of the first C_1', and put an extra level on top of the second copy of C_1' of length equal to the length of the first level of C_1'. This defines C_1 of height $h_1 = 4h_0 + 1$. Now given column C_n of height h_n, obtain column C_{n+1} by cutting each level of C_n in the proportion $\lambda/(1+\lambda)$, $1/(1+\lambda)$ to obtain C_{n+1}'. Then place a copy of C_{n+1}' on top of the first C_{n+1}', and put a level on top of the second C_{n+1}' of length equal to the first level of C_{n+1}'. This defines C_{n+1} of height $h_{n+1} = 4h_n + 1$. On each level the transformation is defined by the unique affine map that takes that level onto the one on top; the contraction or expansion on the level is the value of the Radon–Nikodym derivative on that level. For example, the Radon–Nikodym derivative on the first two levels of C_1 has values $1/\lambda$ and λ. It is clear that the resulting transformation T_λ is conservative ergodic and the contrations or expansions are powers of λ.

THEOREM 4.1: *The transformation T_λ is of type III$_\lambda$, has continuous L^∞ spectrum, all its k-fold cartesian products are conservative and $T_\lambda \times T_\lambda$ is not ergodic.*

Proof: We first show it is type III$_\lambda$. Note that $\omega_{T_\lambda^n}(x) \in \{\lambda^n\}$ and thus $r(T_\lambda) \in \{\lambda^n\} \cup \{0\}$. It suffices to show that $1/\lambda \in r(T_\lambda)$. Let A be a set of positive measure; since the levels generate, there is a level I in some column C_n such that

$$\mu(A \cap I) > \frac{\frac{2}{3}\lambda + 1}{1 + \lambda}\mu(I).$$

Level I is broken into two sublevels I_1 and I_2 in the ratios $\lambda/(1+\lambda)$, $1/(1+\lambda)$ that are used to construct column C_{n+1}. Let k be the first integer such that $T_\lambda^k(I_1) = I_2$. Since T_λ is linear on levels of C_{n+1} and

$$\frac{\mu(I_2)}{\mu(I_1)} = \frac{1}{\lambda}$$

we have $\omega_{T_\lambda^k}(x) = 1/\lambda$ for a.e. $x \in I_1$. Since A covers more than $2/3$ of I_1 and I_2, there is a set of positive measure of $x \in (A \cap I_1) \cap T^{-k}(A \cap I_2)$ with $\omega_{T_\lambda^k}(x) = 1/\lambda$. This shows $1/\lambda \in r(T_\lambda)$.

The proof in Proposition 1.1 applies with minor modifications to show that T_λ has continuous spectrum. In fact, choose the level I so that

$$\mu(A \cap I) > \frac{\frac{2}{3}\lambda^2 + 2\lambda + 1}{(1+\lambda)^2}\mu(I).$$

Then partition I into four subintervals of lengths in the ratios

$$\frac{\lambda^2}{(1+\lambda)^2}, \frac{\lambda}{(1+\lambda)^2}, \frac{\lambda}{(1+\lambda)^2}, \frac{1}{(1+\lambda)^2}.$$

Finally proceed as in the proof of Proposition 1.1.

To show conservativity of the cartesian products we show that T_λ is partially rigid and apply Corollary 1.4. Let I be a level in a column C_n. From the construction of T_λ,

$$\mu(T_\lambda^{h_n} I \cap I) = \frac{1}{1+\lambda}\mu(I).$$

To prove the bound on the Radon–Nikodym derivatives we observe that given a level I in C_n, if I_1 and I_2 are the subintervals of I as above, $\omega_{T_\lambda^{h_n}}(x) = 1/\lambda$ for a.e. $x \in I_1$ and $\omega_{T_\lambda^{h_n}}(x) = \lambda$ for a.e. $x \in I_2$. Lemma 1.2 gives that T_λ is partially rigid.

Finally, one can check that the proof of 1.5 also applies in this case since it only uses the combinatorial structure of the transformation and nonsingularity. ∎

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